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Some Hereditary Properties of WT-Systems

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Let U be the WT-space spanned by a WT-system $\{u_0,..., u_{n-1}\}$ on (a, b). A function f is generalized convex with respect to $\{u_0,..., u_{n-1}\}$ if either $f \in U$ or $\{u_0,..., u_{n-1}, f\}$ is a WT-system. We identity two rather general conditions ("connectedness" and "joinedness") that, when satisfied by U, guarantee the transmittal of certain properties enjoyed by the elements of U to such an f. We characterize WT-spaces that satisfy these conditions and provide equivalent formulations. Various examples are given and applications to spline functions are considered. The main results are as follows: If U is connected then vanishing points of U are zeros of f. If $U \subset C(a, b)$ is connected and joined (i.e., "perfect") then $f \in C(a, b)$. If U', the space of derivatives, is a WT-subspace of $C(a, b)$ then $f \in C^1(a, b)$ provided U' is perfect or U and U' are joined. If $U \subset C(a, b)$ is perfect then the convergence of a sequence $\{f_k\}$ of generalized convex functions to $f \in C(a, b)$ with respect to a fairly inclusive class of seminorms implies its uniform convergence on compact subsets of (a, b). If U and U' are perfect and continuous then this result extends to the sequence of derivatives $\{f_k'\}$ as well.

INTRODUCTION

A set $\{u_0,..., u_{n-1}\}$ of real-valued functions defined on a real interval is called a WT-system (for "weak Tchebysheff") if it is linearly independent and for all points $x_0 < \cdots < x_{n-1}$ in its domain $\det\{u_i(x_j)\}_{i,j=0}^{n-1} \geqslant 0$. The linear span of such a system is called a WT-space. The best known example (at least among numerical analysts) of a WT-system is furnished by the polynomial splines, although the set $\{1, x, f\}$ forms a WT-system whenever f is convex [6, 13]. Two useful references on WT-systems are [4, 12]. Were strict inequality to prevail in the above determinants we would speak of a "T-system." As T-systems are particularly advantageous for numerical applications [2], investigations into the qualitative differences between T-systems and WT-systems have been undertaken in recent years [see, e.g., 8, 10, 11]. These studies show that when the domain is an open interval WT-

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spaces are distinguished from T-spaces solely by the presence either of functions that vanish on subintervals ("degeneracy") or of points in which every element of the space vanishes ("vanishing points"). Functions that are "generalized convex" with respect to a T-system, that is, that can be adjoined to form a WT-system, possess many of the properties of the classical convex functions. In an effort to extend these results to WT-systems, the author was led to the notion of "endpoint nondegeneracy." A WT-space is endpoint nondegenerate (END) if it contains no functions that vanish on a subinterval extending to an endpoint [13, 14; although END is not explicitly defined there]. Unfortunately, this notion is too restrictive as it excludes spaces of spline functions. Two, more fundamental, properties ("connectedness" and "joinedness") are therefore introduced in this paper in order to overcome this unpleasant situation. Although one property was identified to deal with the transmittal of vanishing points to generalized convex functions and the other to handle continuity (we note, though, that END implies both connected and joined), unexpected relations between the two concepts emerged. In Section 1 we introduce these two notions and present some basic consequences. Differentiability is treated in Section 2, and Section 3 is devoted to several further properties of connected and joined WT systems. Examples are deferred to Section 4.

1. Basic Results

Throughout this paper $\{u_0,..., u_{n-1}\}$ and $\{u_0,..., u_n\}$ will denote WT-systems on an open interval (a, b), and U will stand for $\mathrm{sp}\{u_i\}_0^{n-1}$. Continuity is not assumed unless explicitly stated. For points $x_0,..., x_{n-1} \in (a, b)$ we denote

$$U\begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{n-1} \end{pmatrix} = \det\{u_i(x_j)\}_{i,j=0}^{n-1}.$$

Thus, for all $x_0 < \cdots < x_{n-1} < x_n$, we have

$$U\begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{n-1} \end{pmatrix} \geqslant 0 \qquad \text{and} \qquad U\begin{pmatrix} 0,..., n \\ x_0,..., x_n \end{pmatrix} \geqslant 0.$$

When no confusion is likely to result we will occasionally write

$$U\begin{pmatrix} 0,..., n-1 \\ x_0,..., \hat{x}_k,..., x_n \end{pmatrix} \quad \text{to denote} \quad \det\{u_i(x_j)\}_{(i=0 \text{ to } n-1,\, j=0 \text{ to } n,\, j\neq k)}.$$

(1.1) Definition. $\{x_0,..., x_{n-1}\} \subset (a, b)$ is called a *T-set* for $\{u_0,..., u_{n-1}\}$ if $x_0 < \cdots < x_{n-1}$ and $U\binom{0,\ldots,n-1}{x_0,\ldots,x_{n-1}} > 0$.

(1.2) Definition. $\xi \in (a, b)$ is called an *essential point* for U if $u(\xi) \neq 0$ for some $u \in U$. Otherwise ξ is called a *vanishing point.*

Lemma 1.3 combines several elementary results from linear algebra in terms of (1.2).

(1.3) Lemma. *Let A be a set with at least n points.*

(a) *$u_0,..., u_{n-1}$ are linearly independent on A iff* $\det\{u_i(x_j)\}_0^{n-1} \neq 0$ *for some $x_0,..., x_{n-1} \in A$.*

(b) *If $u_0,..., u_{n-1}$ are linearly independent on A then $\xi \in A$ is a vanishing point for* $\mathrm{sp}\{u_i\}_0^{n-1}$ *iff*

$$U\begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{j-1}, \xi, x_{j+1},..., x_{n-1} \end{pmatrix} = 0 \qquad (j = 0 \text{ to } n-1),$$

for some $\{x_0,..., x_{n-1}\} \subset A$ such that $\det\{u_i(x_j)\}_0^{n-1} \neq 0$.

(c) *If $u_0,..., u_{n-1}$ are linearly independent on A and $\xi \in A$ is essential then there are points $x_0,..., x_{n-1} \in A$, including ξ, for which* $\det\{u_i(x_j)\}_0^{n-1} \neq 0$.

(1.4) Definition. (a) $\{u_0,..., u_{n-1}\}$ (or U) is called *connected* if for every vanishing point $\xi \in (a, b)$ there are points $x_0 < \cdots < x_{l-1} < \xi < x_l < \cdots < x_n$ such that $\{x_0,..., x_{j-1}, x_{j+1},..., x_n\}$ is a T-set for some $0 \leqslant j \leqslant l-1$, and $\{x_0,..., x_{k-1}, x_{k+1},..., x_n\}$ is a T-set for some $l \leqslant k \leqslant n$.

(b) $\{u_0,..., u_{n-1}\}$ (or U) is called *joined* if $n \geqslant 2$ and, for every essential point $\xi \in (a, b)$, there are points $x_0 < \cdots < x_{l-1} < \xi = x_l < x_{l+1} < \cdots < x_n$ such that $\{x_0,..., x_{j-1}, x_{j+1},..., x_n\}$ is a T-set for some $0 \leqslant j \leqslant l-1$ and $\{x_0,..., x_{k-1}, x_{k+1},..., x_n\}$ is a T-set for some $l+1 \leqslant k \leqslant n$.

(c) $\{u_0,..., u_{n-1}\}$ (or U) is called *perfect* if it is both connected and joined.

Our first theorem concerns the transmittal of vanishing points to generalized convex functions such as u_n.

Theorem. *If $\{u_0,..., u_{n-1}\}$ is connected and ξ is a vanishing point for U then $u_n(\xi) = 0$.*

Proof. Let $x_0,..., x_n$ be points in (a, b) that satisfy (1.4a). Since ξ is a vanishing point we have

$$0 \leqslant U\begin{pmatrix} 0,..., n \\ x_0,..., \hat{x}_j,..., \xi, x_l,..., x_n \end{pmatrix}$$

$$= (-1)^{n-l+1} u_n(\xi) \cdot U\begin{pmatrix} 0,..., n-1 \\ x_0,..., \hat{x}_j,..., x_n \end{pmatrix},$$

hence,

$$(-1)^{n-l+1}u_n(\xi) \geqslant 0. \tag{1.6}$$

Moreover,

$$0 \leqslant U\begin{pmatrix} 0,\dots, n \\ x_0,\dots, x_{l-1}, \xi, x_l,\dots, \hat{x}_k,\dots, x_n \end{pmatrix}$$
$$= (-1)^{n-l}u_n(\xi) \cdot U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, \hat{x}_k,\dots, x_n \end{pmatrix},$$

from which we deduce that

$$(-1)^{n-l}u_n(\xi) \geqslant 0. \tag{1.7}$$

Theorem 1.5 now follows from (1.6) and (1.7). ∎

(1.8) DEFINITION. $\{u_0,\dots, u_{n-1}\}$ is called a *complete WT-system* if $\{u_0,\dots, u_i\}$ is a WT-system for $i = 0$ to $n-1$.

(1.9) COROLLARY. *If $\{u_0,\dots, u_{n-1}\}$ is a complete* WT-*system on (a, b) and $\{u_0,\dots, u_i\}$ is connected for $i = 0$ to $n-2$ then the vanishing points for U are precisely the zeros of u_0. Consequently, if such a U has no vanishing points then it contains a positive element.*

(1.10) THEOREM. *The following are equivalent*:

(a) *$\{u_0,\dots, u_{n-1}\}$ is not connected ("disconnected").*

(b) *there exists a point $\xi \in (a, b)$ and an $0 \leqslant l \leqslant n-1$ such that $\{x_0,\dots, x_{n-1}\}$ is a* T-*set for $\{u_0,\dots, u_{n-1}\}$ only if $x_{l-1} < \xi < x_l$,*

(c) *there is a* WT-*system $\{v_0,\dots, v_{n-1}\} \subset U$, a vanishing point $\xi \in (a, b)$ and $0 \leqslant l \leqslant n-1$ such that $v_0,\dots, v_{l-1} \equiv 0$ on $[\xi, b)$ and $v_l,\dots, v_{n-1} \equiv 0$ on $(a, \xi]$.*

Proof. (a) ⇒ (c). Suppose that $\{u_0,\dots, u_{n-1}\}$ is disconnected. Then by (1.4a), there is a vanishing point ξ such that for all $x_0 < \cdots < x_{l-1} < \xi < x_l < \cdots < x_n$, all $0 \leqslant j \leqslant l-1$ and $l \leqslant k \leqslant n$, one of $\{x_0,\dots, x_{j-1}, x_{j+1},\dots, x_n\}$ or $\{x_0,\dots, x_{k-1}, x_{k+1},\dots, x_n\}$ is not a T-set. By (1.3a) some T-set $\{x_0,\dots, x_{n-1}\}$ exists, and we assume that $x_{l-1} < \xi < x_l$, where $0 \leqslant l \leqslant n-1$. For $j = 0$ to $n-1$ define

$$v_j(x) = U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{j-1}, x, x_{j+1},\dots, x_{n-1} \end{pmatrix} \Big/ U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-1} \end{pmatrix}.$$

Since $v_j(x_i)=\delta_{ij}$ ($i, j=0$ to $n-1$) the v_j's are linearly independent, hence a basis for U. Moreover, if A is the (nonsingular) matrix corresponding to this change of basis, then for all $y_0,\ldots, y_{n-1} \in (a, b)$,

$$U\begin{pmatrix} 0,\ldots, n-1 \\ y_0,\ldots, y_{n-1} \end{pmatrix} = \det A^{-1} \cdot V\begin{pmatrix} 0,\ldots, n-1 \\ y_0,\ldots, y_{n-1} \end{pmatrix}.$$

By substituting the points $x_0,\ldots, x_{n-1}$ just defined we may deduce that $\{v_0,\ldots, v_{n-1}\}$ is a WT-system. Further, by our choice of $x_0,\ldots, x_{n-1}$ and ξ, we must have $v_0,\ldots, v_{l-1} \equiv 0$ on $[\xi, b)$ and $v_l,\ldots, v_{n-1} \equiv 0$ on $(a, \xi]$. This proves (c).

(c) $\Rightarrow$ (b). Clearly $\{u_0,\ldots, u_{n-1}\}$ and $\{v_0,\ldots, v_{n-1}\}$ have the same T-sets. Let $y_0 < \cdots < y_{n-1}$; then a direct calculation shows that

$$\begin{aligned} V&\begin{pmatrix} 0,\ldots, n-1 \\ y_0,\ldots, y_{n-1} \end{pmatrix} \\ &= V\begin{pmatrix} 0,\ldots, l-1 \\ y_0,\ldots, y_{l-1} \end{pmatrix} \cdot V\begin{pmatrix} l,\ldots, n-1 \\ y_l,\ldots, y_{n-1} \end{pmatrix} \qquad \text{if} \quad y_{l-1} < \xi < y_l, \\ &= 0 \qquad \text{otherwise;} \end{aligned}$$

thus (b) holds.

(b) $\Rightarrow$ (a) is immediate from (1.4a). ∎

As with connectedness, joinedness guarantees the transmittal of a property to all generalized convex functions, in this case continuity.

(1.11) THEOREM. *If $u_0,\ldots, u_{n-1}$ are continuous and $\{u_0,\ldots, u_{n-1}\}$ is joined then u_n is continuous at every essential point; if $\{u_0,\ldots, u_{n-1}\}$ is connected then u_n is continuous at every vanishing point.*

Proof. Let $\xi \in (a, b)$ be an essential point and let $x_0 < \cdots < x_{l-1} < \xi = x_l < x_{l+1} < \cdots < x_n$ be the points guaranteed by (1.4b). Let $\{y^{(\nu)}\}$ be a sequence in (a, b) converging to ξ. We assume that $y^{(\nu)} \uparrow \xi$; then for ν large enough that $x_{l-1} < y^{(\nu)} < \xi$ we have

$$\begin{aligned} 0 &\leqslant U\begin{pmatrix} 0,\ldots, n-1, n \\ x_0,\ldots, \hat{x}_j,\ldots, x_{l-1}, y^{(\nu)}, \xi, x_{l+1},\ldots, x_n \end{pmatrix} \\ &= (-1)^{n-l} \left\{ u_n(\xi) \cdot U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, \hat{x}_j,\ldots, x_{l-1}, y^{(\nu)}, x_{l+1},\ldots, x_n \end{pmatrix} \right. \\ &\quad \left. - u_n(y^{(\nu)}) \cdot U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, \hat{x}_j,\ldots, x_n \end{pmatrix} \right\} \end{aligned}$$

$$+ \sum_{\substack{i=l+1 \text{ to } n \\ \text{and } 0 \text{ to } j-1}} (-1)^{n-i} u_n(x_i) \cdot U \begin{pmatrix} 0,..., n-1 \\ x_0,..., \hat{x}_j,..., y^{(\nu)},..., \hat{x}_i,..., x_n \end{pmatrix}$$

$$+ \sum_{\substack{i=j+1 \\ \text{to } l-1}} (-1)^{n-i-1} u_n(x_i) \cdot U \begin{pmatrix} 0,..., n-1 \\ x_0,..., \hat{x}_i,..., \hat{x}_j,..., y^{(\nu)},..., x_n \end{pmatrix}.$$

where $\{x_0,..., x_{j-1}, x_{j+1},..., x_n\}$ is a T-set for $\{u_0,..., u_{n-1}\}$. By continuity of $u_0,..., u_{n-1}$, all but the first two of these terms vanish as $y^{(\nu)} \uparrow \xi$, hence we may conclude that $\underline{\lim}_{\nu\to\infty}(-1)^{n-l}(u_n(\xi) - u_n(y^{(\nu)})) \geqslant 0$.

A similar argument utilizing the T-set $\{x_0,..., x_{k-1}, x_{k+1},..., x_n\}$ $(l+1 \leqslant k \leqslant n)$ yields $\overline{\lim}_{\nu\to\infty}(-1)^{n-l-1}(u_n(\xi) - u_n(y^{(\nu)})) \geqslant 0$, i.e. $\overline{\lim}_{\nu\to\infty}(-1)^{n-l}(u_n(\xi) - u_n(y^{(\nu)})) \leqslant 0$. Thus, $\overline{\lim}_{\nu\to\infty}(-1)^{n-l}(u_n(\xi) - u_n(y^{(\nu)})) \leqslant 0 \leqslant \underline{\lim}_{\nu\to\infty}(-1)^{n-l}(u_n(\xi) - u_n(y^{(\nu)}))$, and hence $u_n(y^{(\nu)}) \to u_n(\xi)$. This proves the continuity of u_n at every essential point ξ.

The proof that u_n is continuous at every vanishing point of U, under the assumption that U is connected, is carried out in much the same fashion. We assume that ξ is a vanishing point and consider points $x_0 < \cdots < x_{l-1} < \xi < x_l < \cdots < x_n$ guaranteed by the assumption that U is connected. Then for the T-set $\{x_0,..., x_{j-1}, x_{j+1},..., x_n\}$, we write

$$0 \leqslant U \begin{pmatrix} 0,..., n-1, n \\ x_0,..., \hat{x}_j,..., x_{l-1}, y^{(\nu)}, x_l,..., x_n \end{pmatrix}$$

$$= (-1)^{n-l+1} u_n(y^{(\nu)}) \cdot U \begin{pmatrix} 0,..., n-1 \\ x_0,..., \hat{x}_j,..., x_n \end{pmatrix} + o(1) \text{ as } y^{(\nu)} \to \xi.$$

which implies that $\overline{\lim}(-1)^{n-l} u_n(y^{(\nu)}) \leqslant 0$. By considering the T-set $\{x_0,..., x_{k-1}, x_{k+1},..., x_n\}$ $(l \leqslant k \leqslant n)$, we deduce that $\underline{\lim}(-1)^{n-l} u_n(y^{(\nu)}) \geqslant 0$, hence, $u_n(y^{(\nu)}) \to 0$. Since by (1.5) $u_n(\xi) = 0$, this proves the continuity of u_n at each vanishing point of U. ∎

In the remainder of this section we undertake the task of characterizing joined WT-spaces. An initial step in this direction is Theorem 1.12.

(1.12) Theorem. *If $\{u_0,..., u_n\}$ is joined then the vanishing points of U are transmitted to u_n.*

Proof. Suppose that U has a vanishing point ξ such that $u_n(\xi) \neq 0$. Since $\{u_0,..., u_n\}$ is joined, there are points $x_0 < \cdots < x_{l-1} < \xi = x_l < x_{l+1} < \cdots < x_{n+1}$, a $0 \leqslant j \leqslant l-1$ and $l+1 \leqslant k \leqslant n+1$, such that $\{x_0,..., x_{j-1}, x_{j+1},..., x_{n+1}\}$ and $\{x_0,..., x_{k-1}, x_{k+1},..., x_{n+1}\}$ are T-sets. But then

$$0 < U\begin{pmatrix} 0,\ldots, n-1, n \\ x_0,\ldots, \hat{x}_j,\ldots, x_{n+1} \end{pmatrix}$$

$$= (-1)^{n-l+1} u_n(\xi) \cdot U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, \hat{x}_j,\ldots, \hat{x}_l,\ldots, x_{n+1} \end{pmatrix},$$

which implies $(-1)^{n-l+1}u_n(\xi) > 0$, and

$$0 < U\begin{pmatrix} 0,\ldots, n-1, n \\ x_0,\ldots, \hat{x}_k,\ldots, x_{n+1} \end{pmatrix}$$

$$= (-1)^{n-l} u_n(\xi) \cdot U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, \hat{x}_l,\ldots, \hat{x}_k,\ldots, x_{n+1} \end{pmatrix},$$

which implies $(-1)^{n-l}u_n(\xi) > 0$, a contradiction. ∎

(1.13) COROLLARY. *If $\{u_0,\ldots, u_{n-1}\}$ is a complete* WT-*system such that $\{u_0,\ldots, u_i\}$ is joined for $i = 1,\ldots, n-1$ then the vanishing points for U are precisely the zeros of u_0. Thus if such a U has no vanishing points then it contains a positive function.*

(1.14) *Remark.* It is evident from the proofs of the preceding results that (1.4a) and (1.4b) are not the weakest possible definitions that would ensure these results. For instance, to guarantee the transmittal of a vanishing point ξ to u_n it suffices to find *two* T-sets $x_0 < \cdots < x_{j-1} < \xi < x_j < \cdots < x_{n-1}$ and $y_0 < \cdots < y_{i-1} < \xi < y_i < \cdots < y_{n-1}$ such that $i + j$ is odd. This is clearly a weaker condition than (1.4a).

(1.15) THEOREM. *U is joined iff the vanishing points of all $(n-1)$-dimensional* WT-*subspaces are vanishing points for U.*

Proof. If U is joined then by (1.12) the vanishing points of all $(n-1)$-dimensional WT-subspaces are transmitted to U. To prove the converse, assume that U is not joined ("disjoined"). Then there is an essential point $\xi \in (a, b)$ such that for all $x_0 < \cdots < x_{l-1} < \xi = x_l < x_{l+1} < \cdots < x_n$, all $0 \leqslant j \leqslant l-1$ and $l+1 \leqslant k \leqslant n$, one of $\{x_0,\ldots, x_{j-1}, x_{j+1},\ldots, x_n\}$ or $\{x_0,\ldots, x_{k-1}, x_{k+1},\ldots, x_n\}$ is not a T-set. Since ξ is essential (1.3c) guarantees that a T-set $\{x_0,\ldots, x_{n-1}\}$, with $\xi = x_l$ for some $0 \leqslant l \leqslant n-1$, exists. We now define $v_0,\ldots, v_{n-1}$ as in (1.10) and note that our choice of ξ requires that $v_0,\ldots, v_{l-1} \equiv 0$ on $[\xi, b)$ and $v_{l+1},\ldots, v_{n-1} \equiv 0$ on $(a, \xi]$. Concerning v_l, we can only state that $v_l(\xi) = 1$. For arbitrary points $y_0 < \cdots < y_{n-1}$ one may easily check that

$$\begin{aligned}\det\{v_i(y_j)\}_{i,j=0}^{n-1} &= V\begin{pmatrix}0,\dots,n-1\\ y_0,\dots,y_{n-1}\end{pmatrix}\\ &= V\begin{pmatrix}0,\dots,l-1\\ y_0,\dots,y_{l-1}\end{pmatrix}\cdot V\begin{pmatrix}l,\dots,n-1\\ y_l,\dots,y_{n-1}\end{pmatrix}\\ &\qquad\qquad \text{if}\quad y_{l-1}<\xi<y_{l+1} \text{ and } y_l\leqslant\xi,\\ &= V\begin{pmatrix}0,\dots,l\\ y_0,\dots,y_l\end{pmatrix}\cdot V\begin{pmatrix}l+1,\dots,n-1\\ y_{l+1},\dots,y_{n-1}\end{pmatrix}\\ &\qquad\qquad \text{if}\quad y_{l-1}<\xi<y_{l+1} \text{ and } y_l\geqslant\xi,\\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{1.16}$$

By substituting $x_l,\dots,x_{n-1}$ for $y_l,\dots,y_{n-1}$ in (1.16) we get $V(\begin{smallmatrix}l,\dots,n-1\\ x_l,\dots,x_{n-1}\end{smallmatrix})=1$, hence we may conclude that $V(\begin{smallmatrix}0,\dots,l-1\\ y_0,\dots,y_{l-1}\end{smallmatrix})\geqslant 0$ for all $y_0<\cdots<y_{l-1}$ in $(a,\xi]$ (for $y_{l-1}=\xi$ the determinant is zero). This implies that $\{v_0,\dots,v_{l-1}\}$ is a WT-system on $(a,\xi]$. Similarly, we deduce that $\{v_{l+1},\dots,v_{n-1}\}$ is a WT-system on $[\xi,b)$. We now demonstrate that $\{v_0,\dots,v_{l-1},v_{l+1},\dots,v_{n-1}\}$ is a WT-system on (a,b). Indeed, since $v_0,\dots,v_{l-1}\equiv 0$ on $[\xi,b)$ and $v_{l+1},\dots,v_{n-1}\equiv 0$ on $(a,\xi]$, for any $z_0<\cdots<z_{n-2}$ we have

$$\begin{aligned}&V\begin{pmatrix}0,\dots,l-1,l+1,\dots,n-1\\ z_0,\dots,z_{n-2}\end{pmatrix}\\ &\quad= V\begin{pmatrix}0,\dots,l-1\\ z_0,\dots,z_{l-1}\end{pmatrix}\cdot V\begin{pmatrix}l+1,\dots,n-1\\ z_l,\dots,z_{n-2}\end{pmatrix} \qquad \text{if}\quad z_{l-1}<\xi<z_l,\\ &\quad= 0 \qquad \text{otherwise.}\end{aligned} \tag{1.17}$$

Since we have shown that each of the determinants in the product (1.17) is nonnegative, this proves that $\{v_0,\dots,v_{l-1},v_{l+1},\dots,v_{n-1}\}$ is a WT-system. Moreover, the vanishing point ξ is not transmitted to U. ∎

(1.18) *Remark.* From the proof of (1.15) we may infer that U is disjoined iff there is an essential point ξ and $0\leqslant l\leqslant n-1$ such that $\{x_0,\dots,x_{n-1}\}$ is a T-set for U only if $x_{l-1}<\xi<x_{l+1}$.

(1.19) Corollary. (a) *If all $(n-1)$-dimensional* WT-*subspaces of U are connected then U is joined.*

(b) *If no $(n-1)$-dimensional* WT-*subspaces of U have vanishing points then U is joined.*

Proof. Both (a) and (b) imply that the vanishing points of all WT-subspaces are transmitted to U, hence, the conclusions follow from (1.15). ∎

2. Differentiability

In this section we investigate conditions under which differentiability is inherited by generalized convex functions. We therefore assume henceforth that the functions $u_0,\dots,u_{n-1}$ are continuously differentiable in (a,b). In addition, we restrict our attention to WT-spaces U such that U', the space of derivatives, is also a WT-space. Such WT-spaces have been characterized in [15]:

(2.1) Theorem. *If U is a finite dimensional linear space of continuously differentiable functions, containing constants, then U' is a* WT-*space iff U has a basis $\{u_0,\dots,u_{n-1}\}$ $(n \geqslant 2)$ that is a complete* WT-*system with $u_0 \equiv 1$. If this is the case then $\{u_1',\dots,u_{n-1}'\}$ is itself a complete* WT-*system.*

(2.2) Lemma. *Let $\{u_0,\dots,u_{n-1}\}$ $(n \geqslant 2)$ be a complete* WT-*system with $u_0 \equiv 1$. For points $x_0 < \cdots < x_{n-1}$,*

$$U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{n-1} \end{pmatrix} = 0 \qquad \text{iff} \qquad U'\begin{pmatrix} 1,\dots,n-1 \\ \eta_1,\dots,\eta_{n-1} \end{pmatrix} = \det\{u_i'(\eta_j)\}_{i,j=1}^{n-1} = 0,$$

for all $x_{i-1} < \eta_i < x_i$ $(i = 1$ to $n-1)$.

Proof.

$$U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{n-1} \end{pmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ u_1(x_0) & u_1(x_1)-u_1(x_0) & \cdots & u_1(x_{n-1})-u_1(x_{n-2}) \\ \vdots & \vdots & & \vdots \\ u_{n-1}(x_0) & u_{n-1}(x_1)-u_{n-1}(x_0) & \cdots & u_{n-1}(x_{n-1})-u_{n-1}(x_{n-2}) \end{vmatrix}$$

$$= \begin{vmatrix} u_1(_1)-u_1(x_0) & \cdots & u_1(x_{n-1})-u_1(x_{n-2}) \\ \vdots & & \vdots \\ u_{n-1}(x_1)-u_{n-1}(x_0) & \cdots & u_{n-1}(x_{n-1})-u_{n-1}(x_{n-2}) \end{vmatrix}$$

$$= \int_{x_0}^{x_1}\int_{x_1}^{x_2}\cdots\int_{x_{n-2}}^{x_{n-1}} \begin{vmatrix} u_1'(\eta_1) & \cdots & u_1'(\eta_{n-1}) \\ \vdots & & \vdots \\ u_{n-1}'(\eta_1) & \cdots & u_{n-1}'(\eta_{n-1}) \end{vmatrix} d\eta_{n-1}\cdots d\eta_1. \tag{2.3}$$

Since from (2.1) both $\{u_0,\dots,u_{n-1}\}$ and $\{u_1',\dots,u_{n-1}'\}$ are WT-systems and $u_1',\dots,u_{n-1}'$ are continuous, Lemma 2.2 follows from (2.3). ∎

As with continuity, differentiability requires that we treat vanishing points and essential points separately.

(2.4) THEOREM. *If U' is connected or U is joined, and $n \geqslant 2$, then u_n' exists at each vanishing point of U' and vanishes there.*

Proof. Let ξ be a vanishing point for U'. If U' is connected then there are points $\eta_1 < \cdots < \eta_l < \xi < \eta_{l+1} < \cdots < \eta_n$ such that $\{\eta_1,\dots,\eta_{j-1},\eta_{j+1},\dots,\eta_n\}$ is a T-set for U' for some $1 \leqslant j \leqslant l$, and $\{\eta_1,\dots,\eta_{k-1},\eta_{k+1},\dots,\eta_n\}$ is a T-set for some $l+1 \leqslant k \leqslant n$. For $x_0,\dots,x_n$ with $x_0 < \eta_1 < x_1 < \cdots < \eta_l < x_l = \xi < \eta_{l+1} < \cdots < \eta_n < x_n$, (2.2) indicates that $\{x_0,\dots,x_{j-1},x_{j+1},\dots,x_n\}$ and $\{x_0,\dots,x_{k-1},x_{k+1},\dots,x_n\}$ are T-sets for U. Thus, for $\xi < x < x_{l+1}$,

$$0 \leqslant U\begin{pmatrix} 0,\dots,n-1,n \\ x_0,\dots,x_{j-1},x_{j+1},\dots,\xi,x,x_{l+1},\dots,x_n\end{pmatrix}$$

$$= (x-\xi)\cdot\begin{vmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ u_1(x_0) & \cdots & u_1(\xi) & u_1|\xi,x| & u_1(x_{l+1}) & \cdots & u_1(x_n) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_n(x_0) & \cdots & u_n(\xi) & u_n|\xi,x| & u_n(x_{l+1}) & \cdots & u_n(x_n)\end{vmatrix},$$

where $u_i|\xi,x| = (u_i(x)-u_i(\xi))/(x-\xi)$ $(i=1$ to $n)$. Expanding this determinant, we get $(-1)^{n-l}u_n|\xi,x|\cdot U\binom{0,\dots,n-1}{x_0,\dots,\hat{x}_j,\dots,x_n} + o(1)$ as $x \downarrow \xi$, since $u_i'(\xi)=0$ $(i=1$ to $n-1)$. Hence, $\underline{\lim}_{x\downarrow\xi}(-1)^{n-l}u_n|\xi,x| \geqslant 0$. By utilizing the T-set $\{x_0,\dots,x_{k-1},x_{k+1},\dots,x_n\}$ we may infer that $-\overline{\lim}_{x\downarrow\xi}(-1)^{n-l}u_n|\xi,x| = \underline{\lim}_{x\downarrow\xi}(-1)^{n-l-1}u_n|\xi,x| \geqslant 0$, hence,

$$\underline{\lim}_{x\downarrow\xi}(-1)^{n-l}u_n|\xi,x| \geqslant 0 \geqslant \overline{\lim}_{x\downarrow\xi}(-1)^{n-l}u_n|\xi,x|,$$

which implies that $\lim_{x\downarrow\xi}u_n|\xi,x| = 0$. An analogous argument shows that $\lim_{x\uparrow\xi}u_n|\xi,x|=0$, so that $u_n'(\xi)$ exists and is equal to zero.

If U is joined then, since ξ is an essential point for U, there exist points $y_0 < \cdots < y_{l-1} < \xi = y_l < \cdots < y_n$, a $0 \leqslant j \leqslant l-1$ and $l+1 \leqslant k \leqslant n$ such that $\{y_0,\dots,y_{j-1},y_{j+1},\dots,y_n\}$ and $\{y_0,\dots,y_{k-1},y_{k+1},\dots,y_n\}$ are T-sets for U. We may now proceed as before to show that $u_n'(\xi)$ exists and is zero. ∎

(2.5) DEFINITION. We will say that $\{x_0,\dots,x_{l-2},\xi,\xi,x_{l+1},\dots,x_{n-1}\}$ is a T*-*set* for U if $x_0 < \cdots < x_{l-2} < \xi < x_{l+1} < \cdots < x_{n-1}$ and

$$0 < U^*\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{l-2},\xi,\xi,x_{l+1},\dots,x_{n-1}\end{pmatrix}$$

$$= \begin{vmatrix} u_0(x_0) & \cdots & u_0(\xi) & u_0'(\xi) & u_0(x_{l+1}) & \cdots & u_0(x_{n-1}) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ u_{n-1}(x_0) & \cdots & u_{n-1}(\xi) & u_{n-1}'(\xi) & u_{n-1}(x_{l+1}) & \cdots & u_{n-1}(x_{n-1})\end{vmatrix}.$$

In order to prove a result similar to (2.4) about the essential points of U', we need the next two lemmas.

(2.6) LEMMA. *Let $u_0,\dots,u_{n-1}$ be linearly independent, continuously differentiable functions with $u_0 \equiv 1$. If $n \geqslant 2$ and ξ is an essential point for* $\operatorname{sp}\{u_1',\dots,u_{n-1}'\}$ *then there exist points $x_0,\dots,x_{n-3}$ such that* $U^*\binom{0,\dots,n-1}{x_0,\dots,x_{n-3},\xi,\xi} \neq 0$.

Proof. We apply induction on n. For $n=2$ we have

$$\begin{vmatrix} u_0(\xi) & u_0'(\xi) \\ u_1(\xi) & u_1'(\xi) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ u_1(\xi) & u_1'(\xi) \end{vmatrix} = u_1'(\xi) \neq 0,$$

since ξ is essential for $\operatorname{sp}\{u_1'\}$. For $n>2$ let $x_0,\dots,x_{n-3}$ be arbitrary points; then

$$U^*\begin{pmatrix} 0,\dots,n-1 \\ x_1,\dots,x_{n-3},\xi,\xi \end{pmatrix}$$
$$= \sum_{i=0}^{n-1} (-1)^{n-1-i} u_i(x_0) \cdot U^*\begin{pmatrix} 0,\dots,i-1,i+1,\dots,n-1 \\ x_1,\dots,x_{n-3},\xi,\xi \end{pmatrix}. \quad (2.7)$$

If for fixed $x_1,\dots,x_{n-3}$, (2.7) is identically zero as a function of x_0 then by the linear independence of $u_0,\dots,u_{n-1}$ we must have $U^*\binom{0,\dots,i-1,i+1,\dots,n-1}{x_1,\dots,x_{n-3},\xi,\xi} = 0$ $(i=0$ to $n-1)$. Were this to hold for all $x_1,\dots,x_{n-3}$, the induction hypothesis would imply that ξ is a vanishing point for $\operatorname{sp}\{u_1',\dots,u_{i-1}', u_{i+1}',\dots,u_{n-1}'\}$ $(i=1$ to $n-1)$ and hence for $\operatorname{sp}\{u_1',\dots,u_{n-1}'\}$. ∎

(2.8) LEMMA. *If there exist points $x_0,\dots,x_{l-2}$, $x_{l+1},\dots,x_n$, such that $\{x_0,\dots,x_{j-1}, x_{j+1},\dots,x_{l-2}, \xi, \xi, x_{l+1},\dots,x_n\}$ and $\{x_0,\dots,x_{l-2}, \xi, \xi, x_{l+1},\dots,x_{k-1}, x_{k+1},\dots,x_n\}$ are T*-sets for some $0 \leqslant j \leqslant l-2$ and $l+1 \leqslant k \leqslant n$, then u_n is differentiable at ξ, provided ξ is an essential point for U'.*

Proof. For points $x_{l-2} < x < \xi < y < x_{l+1}$, we have

$$0 \leqslant U\begin{pmatrix} 0,\dots,n \\ x_0,\dots,\hat{x}_j,\dots,x_{l-1},x,\xi,y,x_{l+1},\dots,x_n \end{pmatrix}$$

$$= (x-\xi)\cdot(y-\xi) \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ u_1(x_0) & \cdots & u_1(x_{j-1}) & u_1(x_{j+1}) & \cdots & u_1(x_{l-2}) & u_1[x,\xi] & u_1(\xi) & u_1[\xi,y] & u_1(x_{l+1}) & \cdots & u_1(x_n) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ u_n(x_0) & \cdots & u_n(x_{j-1}) & u_n(x_{j+1} & \cdots & u_n(x_{l-2}) & u_n[x,\xi] & u_n(\xi) & u_n[\xi,y] & u_n(x_{l+1}) & \cdots & u_n(x_n) \end{vmatrix},$$

where $u_i[x, \xi] = (u_i(x) - u_i(\xi))/(x - \xi)$ $(i = 1$ to $n)$. Hence,

$$0 \leqslant \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ u_1(x_0) & \cdots & u_1(x_{j-1}) & u_1(x_{j+1}) & \cdots & u_1(x_{l-2}) \\ \vdots & & \vdots & \vdots & & \vdots \\ u_n(x_0) & \cdots & u_n(x_{j-1}) & u_n(x_{j+1}) & \cdots & u_n(x_{l-2}) \end{matrix}$$
$$\begin{matrix} 1 & 0 & 0 & 1 & \cdots & 1 \\ u_1(\xi) & u_1[x, \xi] & u_1[\xi, y] & u_1(x_{l+1}) & \cdots & u_1(x_n) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ u_n(\xi) & u_n[x, \xi] & u_n[\xi, y] & u_n(x_{l+1}) & \cdots & u_n(x_n) \end{vmatrix}.$$

We now assume without loss of generality that $n - l$ is even (otherwise we replace $\overline{\lim}$ by $\underline{\lim}$ and vice versa). Expanding by the last row and letting first $y \downarrow \xi$ and then $x \uparrow \xi$, we see that the nonvanishing terms satisfy

$$0 \leqslant (-1)^{n-l}\Big(\underline{\lim}_{y \downarrow \xi} u_n[\xi, y] - \overline{\lim}_{x \uparrow \xi} u_n[x, \xi]\Big)$$
$$\cdot\, U^*\begin{pmatrix} 0, \ldots, n-1 \\ x_0, \ldots, \hat{x}_j, \ldots, x_{l-2}, \xi, \xi, x_{l+1}, \ldots, x_n \end{pmatrix},$$

that is, $(-1)^{n-l}\,\underline{\lim}_{y \downarrow \xi}\, u_n[\xi, y] \geqslant (-1)^{n-l}\,\overline{\lim}_{x \uparrow \xi}\, u_n[x, \xi]$. If we employ the same reasoning with respect to the T*-set $\{x_0, \ldots, x_{l-2}, \xi, \xi, x_{l+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\}$ we may conclude that

$$(-1)^{n-l}\,\underline{\lim}_{x \uparrow \xi}\, u_n[x, \xi] \geqslant (-1)^{n-l}\,\overline{\lim}_{y \downarrow \xi}\, u_n[\xi, y].$$

Thus

$$\underline{\lim}_{y \downarrow \xi}\, u_n[\xi, y] \geqslant \overline{\lim}_{x \uparrow \xi}\, u_n[x, \xi] \geqslant \underline{\lim}_{x \uparrow \xi}\, u_n[x, \xi] \geqslant \overline{\lim}_{y \downarrow \xi}\, u_n[\xi, y],$$

implying the equality of each term. This demonstrates the existence and equality of the one-sided derivatives $f'_+(\xi)$ and $f'_-(\xi)$, and thus the differentiability of f at ξ. ∎

We can now extend our differentiability result to the essential points of U'.

(2.9) THEOREM. *If U' is joined $(n \geqslant 3)$ then u_n is continuously differentiable on the set of essential points of U'.*

Proof. Let ξ be an essential point for U'. We show first that if u_n is not differentiable at ξ then U' is disjoined. By (2.6) there exist points $x_0 < \cdots < x_{l-2} < \xi < x_{l+1} < \cdots < x_{n-1}$ such that $\{x_0, \ldots, x_{l-2}, \xi, \xi, x_{l+1}, \ldots, x_{n-1}\}$ is a T*-set for U. Define

$$v_i(x) = U^* \begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{i-1}, x, x_{i+1},..., x_{l-2}, \xi, \xi, x_{l+1},..., x_{n-1} \end{pmatrix}$$

$(i = 0$ to $l-2)$, and

$$v_i(x) = U^* \begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{l-2}, \xi, \xi, x_{l+1},..., x_{i-1}, x, x_{i+1},..., x_{n-1} \end{pmatrix}$$

$(i = l+1$ to $n-1)$; v_{l-1} and v_l can be defined so that $\{v_0,..., v_{n-1}\}$ is a WT-basis for U. Since u_n is not differentiable at ξ, (2.8) implies that $v_0,..., v_{l-2} \equiv 0$ on $[\xi, b)$ and $v_{l+1},..., v_{n-1} \equiv 0$ on $(a, \xi]$, which in turn implies that for points $y_0 < \cdots < y_{n-1}$ the determinant $U(\begin{smallmatrix} 0,...,n-1 \\ y_0,...,y_{n-1} \end{smallmatrix})$ vanishes unless $y_{l-2} < \xi < y_{l+1}$. Now let $\{\eta_1,..., \eta_{n-1}\}$ be a T-set for U'. If $\eta_{l-1} > \xi$ then by (2.2) any points $y_0,..., y_{n-1}$ such that $\xi \leqslant y_{l-2} < \eta_{l-1}$ and $y_{i-1} < \eta_i < y_i$ $(i = 1$ to $n-1)$ are a T-set for U, in contradiction to what we have just obtained. By continuity we may conclude that $U'(\begin{smallmatrix} 1,...,n-1 \\ \eta_1,...,\eta_{n-1} \end{smallmatrix}) = 0$ unless $\eta_{l-1} < \xi$. Similarly, this determinant vanishes if $\eta_{l+1} \leqslant \xi$, so that from (1.18) U' is necessarily disjoined. We have thus shown that if U' is joined then u_n is differentiable on the set of essential points of U'. The continuity of u_n' then follows from (1.11). ∎

Combining (2.4) and (2.9) yields

(2.10) Corollary. *If U' is perfect, or U and U' are joined then u_n is continuously differentiable in (a, b).*

3. Further Results

In this section we present some further results concerning the convergence of sequences of generalized convex functions and their derivatives. Initial results along this line were given in [3, 7].

(3.1) Definition. The *generalized convex cone* $C(u_0,..., u_{n-1})$ of a given WT-system $\{u_0,..., u_{n-1}\}$ consists of $\text{sp}\{u_0,..., u_{n-1}\}$ and all functions f such that $\{u_0,..., u_{n-1}, f\}$ is a WT-system.

$C(u_0,..., u_{n-1})$ is indeed a convex cone and is closed with respect to pointwise convergence of its elements.

The following useful definition was introduced in [3], a paper that inspired much of the material in this section.

(3.2) Definition. A sequence $\{f_k\}$ is said to *converge nearly* to f if for every open interval I and every $\varepsilon > 0$ there is a $K > 0$ such that for every $k \geqslant K$ there is an $x_k \in I$ for which $|f_k(x_k) - f(x_k)| < \varepsilon$.

We note that if $\|\cdot\|$ is any seminorm on a space of functions F with the property that for every $\varepsilon > 0$ and every open interval I there is a $\delta > 0$ such that, for each $f \in F$, $|f(x)| \geqslant \varepsilon$ for all $x \in I \Rightarrow \|f\| \geqslant \delta$, then convergence with respect to the seminorm implies near convergence. Many commonly used seminorms are of this variety, for example, the L_p-norm

$$\|f\| = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} \qquad (1 \leqslant p < \infty).$$

An equivalent, sometimes more convenient, definition for near convergence is provided by Lemma 3.3.

(3.3) LEMMA. *$f_k \to f$ nearly, iff for every $x \in (a, b)$ there is a sequence $x_k \to x$ such that $|f_k(x_k) - f(x_k)| \to 0$.*

Proof. Let I be an open interval, $\varepsilon > 0$, and let $x \in I$. If $x_k \to x$ and $|f_k(x_k) - f(x_k)| \to 0$ then there is a K_1 such that $k \geqslant K_1 \Rightarrow x_k \in I$, and there is a $K_2 \geqslant K_1$ such that $k \geqslant K_2 \Rightarrow |f_k(x_k) - f(x_k)| < \varepsilon$. This proves one direction. Conversely, let $x \in (a, b)$ be given and denote $I_j = (x - (1/j), x + (1/j))$, $\varepsilon_j = 1/j$ for $j \geqslant 1$. For simplicity assume that $I_j \subset (a, b)$ for all j. If $f_k \to f$ nearly, then for each j we can choose a $k(j) > k(j-1)$ such that for all $k \geqslant k(j)$ there is a point $x_k^{(j)} \in I_j$ such that $|f_k(x_k^{(j)}) - f(x_k^{(j)})| < \varepsilon_j$. Define a sequence $\{x_k\}$ as follows: $x_1, \ldots, x_{k(1)-1} = x$, and for all $j \geqslant 1$ $x_{k(j)} = x_{k(j)}^{(j)}$, $x_{k(j)+1} = x_{k(j)+1}^{(j)}, \ldots, x_{k(j+1)-1} = x_{k(j+1)-1}^{(j)}$. Clearly, $x_k \to x$ and $|f_k(x_k) - f(x_k)| \to 0$. ∎

(3.4) *Remark.* If f is continuous then an application of the triangle inequality shows that $f_k \to f$ nearly, iff for every $x \in (a, b)$ there is a sequence $x_k \to x$ such that $|f_k(x_k) - f(x)| \to 0$.

In the following material we will occasionally use the notation

$$U\begin{pmatrix} 0, \ldots, n-1; f \\ x_0, \ldots, x_n \end{pmatrix} = \begin{vmatrix} u_0(x_0) & \cdots & u_0(x_n) \\ \vdots & & \vdots \\ u_{n-1}(x_0) & \cdots & u_{n-1}(x_n) \\ f(x_0) & \cdots & f(x_n) \end{vmatrix}.$$

Note that $f \in \mathrm{C}(u_0, \ldots, u_{n-1})$ iff $U\binom{0, \ldots, n-1; f}{x_0, \ldots, x_n} \geqslant 0$ for all $x_0 < \cdots < x_n$.

(3.5) LEMMA [cf. 3, Theorem 2.1]. *Let $\{u_0, \ldots, u_{n-1}\}$ be a continuous WT-system on (a, b). If $\{f_k\} \subset \mathrm{C}(u_0, \ldots, u_{n-1})$ converges nearly to a continuous function f then $f \in \mathrm{C}(u_0, \ldots, u_{n-1})$.*

Proof. Let $a < x_0 < \cdots < x_n < b$ be given. Since $f_k \to f$ nearly, by (3.4) there are sequences $\{x_j^{(k)}\}_{k=1}^{\infty}$ ($j = 0$ to n) such that $x_j^{(k)} \to x_j$ and $f_k(x_j^{(k)}) \to$

$f(x_j)$. Moreover, by continuity $u_i(x_j^{(k)}) \to u_i(x_j)$ ($i=0$ to $n-1$), hence, for large enough k,

$$0 \leqslant U\begin{pmatrix} 0,..., n-1; f_k \\ x_0,..., x_{n-1}^{(k)}, x_n^{(k)} \end{pmatrix} \xrightarrow[k\to\infty]{} U\begin{pmatrix} 0,..., n-1; f \\ x_0,..., x_{n-1}, x_n \end{pmatrix}.$$

Hence, $f \in C(u_0,..., u_{n-1})$. ∎

(3.6) EXAMPLE. Let $f_k(x) = \sin(kx)$ for $x \in [0, 2\pi]$; f_k vanishes at $x = j\pi/k$ ($j=0$ to $2k$), hence, since the set $\{(j\pi/k): j=0 \text{ to } 2k;\ k=1,...\}$ is dense in $[0, 2\pi]$, $f_k \to 0$ nearly. However, $f_k \nrightarrow 0$ pointwise.

Our next result shows that under certain conditions near convergence does imply pointwise convergence.

(3.7) THEOREM. *Let $\{u_0,..., u_{n-1}\}$ be a perfect* WT-*system of continuous functions on (a, b). If $\{f_k\} \subset C(u_0,..., u_{n-1})$ converges nearly to a continuous function f then $f_k \to f$ pointwise in (a, b).*

Proof. Suppose that $f_k \nrightarrow f$, then there is a point $\xi \in (a, b)$, an $\varepsilon > 0$, and a subsequence, which we relabel $\{f_k\}$, such that

$$|f_k(\xi) - f(\xi)| \geqslant \varepsilon \qquad \text{for all } k. \tag{3.8}$$

By (3.5) $f \in C(u_0,..., u_{n-1})$, hence ξ must be an essential point for U since otherwise (1.5) implies that $f_k(\xi) = f(\xi) = 0$ for all k, contrary to (3.8). Thus, (1.4b) guarantees the existence of points $x_0 < \cdots < x_{l-1} < \xi = x_l < x_{l+1} < \cdots < x_n$ such that $\{x_0,..., x_{j-1}, x_{j+1},..., x_n\}$ and $\{x_0,..., x_{i-1}, x_{i+1},..., x_n\}$ are T-sets for some $0 \leqslant j \leqslant l-1$ and $l+1 \leqslant i \leqslant n$, respectively. Since $\{f_k\}$ is nearly convergent there exist sequences $x_j^{(k)} \xrightarrow[k\to\infty]{} x_j$ ($j=0$ to n) such that

$$|f_k(x_j^{(k)}) - f(x_j^{(k)})| \xrightarrow[k\to\infty]{} 0 \qquad (j=0 \text{ to } n). \tag{3.9}$$

Since ξ is an essential point there must be an element $u \in U$ such that $u(\xi) > 0$. With this u define

$$g_k(x) = f_k(x) - \frac{f(\xi)}{u(\xi)} \cdot u(x),$$

then $g_k(\xi) = f_k(\xi) - f(\xi)$ for all k. We now show that (3.8) leads to a contradiction. It suffices to consider the following cases:

Case 1. $x_l^{(k)} \uparrow \xi$ and $(-1)^{n-l} g_k(\xi) \leqslant -\varepsilon$.

Case 2. $x_l^{(k)} \downarrow \xi$ and $(-1)^{n-l} g_k(\xi) \geqslant \varepsilon$.

Case 3. $x_l^{(k)} \uparrow \xi$ and $(-1)^{n-l} g_k(\xi) \leqslant -\varepsilon$.

Case 4. $x_l^{(k)} \downarrow \xi$ *and* $(-1)^{n-l} g_k(\xi) \geqslant \varepsilon$.

For Case 1 we note that for large enough k

$$0 \leqslant U\begin{pmatrix} 0,\dots, n-1; f_k \\ x_0^{(k)},\dots, \hat{x}_j^{(k)},\dots, x_l^{(k)}, \xi, x_{l+1}^{(k)},\dots, x_n^{(k)} \end{pmatrix}$$

$$= U\begin{pmatrix} 0,\dots, n-1; g_k \\ x_0^{(k)},\dots, \hat{x}_j^{(k)},\dots, x_l^{(k)}, \xi, x_{l+1}^{(k)},\dots, x_n^{(k)} \end{pmatrix}$$

$$= (-1)^{n-l} g_k(\xi) \cdot U\begin{pmatrix} 0,\dots, n-1 \\ x_0^{(k)},\dots, \hat{x}_j^{(k)},\dots, x_n^{(k)} \end{pmatrix} + o(1) \qquad \text{as} \quad k \to \infty.$$

This is a result of the continuity of $u_0,\dots, u_{n-1}$ and (3.9). Since

$$\lim_{k\to\infty} U\begin{pmatrix} 0,\dots, n-1 \\ x_0^{(k)},\dots, \hat{x}_j^{(k)},\dots, x_n^{(k)} \end{pmatrix} = U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, \hat{x}_j,\dots, x_n \end{pmatrix} > 0$$

we may write $0 \leqslant -\varepsilon \cdot U\left(\begin{smallmatrix} 0,\dots, n-1 \\ x_0,\dots, \hat{x}_j,\dots, x_n \end{smallmatrix}\right) + o(1)$, which is clearly a contradiction. The preceding proof carries over verbatim to Case 2 after switching the columns with $x_l^{(k)}$ and ξ in the determinant. In order to treat Cases 3 and 4 we employ the T-set $\{x_0,\dots, x_{i-1}, x_{i+1},\dots, x_n\}$ and proceed as before. ∎

Theorem 3.10 was proved in [13] under more restrictive assumptions on U. However, the proof carries over almost intact with the aid of (1.5) and (1.11).

(3.10) THEOREM. *Under the conditions of* (3.7), *if* $\{f_k\} \subset C(u_0,\dots, u_{n-1})$ *converges pointwise to a function* f *then* $\{f_k\}$ *converges uniformly to* f *on compact subsets of* (a, b).

Combining (3.7) and (3.10), we get

(3.11) COROLLARY. *Let the conditions of* (3.7) *prevail. If* $\{f_k\} \subset C(u_0,\dots, u_{n-1})$ *converges nearly to a continuous function* f *then the convergence is uniform on compact subsets of* (a, b).

We now consider sequences of derivatives, starting with a lemma on near convergence.

(3.12) LEMMA. *Let* $\{g_k\}$ *be a sequence of continuously differentiable functions converging nearly to zero in* (a, b). *Then* $\{g_k'\}$ *converges nearly to zero in* (a, b).

Proof. Suppose that $g_k' \nrightarrow 0$ nearly, then there is an $\varepsilon > 0$, an interval $(\alpha, \beta) \subset (a, b)$, and a subsequence, which we relabel $\{g_k'\}$ such that $|g_k'(x)| \geqslant \varepsilon$ for all k and all $x \in (\alpha, \beta)$. Since g_k' is continuous there is a $\sigma_k = \pm 1$ such that $\sigma_k g_k'(x) \geqslant \varepsilon$ for $x \in (\alpha, \beta)$, and we may assume without losing generality that all $\sigma_k = 1$. Choose $\alpha < x_0 < x_1 < \beta$, then by assumption there are sequences $x_0^{(k)} \to x_0$ and $x_1^{(k)} \to x_1$ such that $g_k(x_i^{(k)}) \to 0$ $(i = 0, 1)$. For k large enough so that $\alpha < x_0^{(k)} < x_1^{(k)} < \beta$ we then have

$$g_k(x_1^{(k)}) - g_k(x_0^{(k)}) = \int_{x_0^{(k)}}^{x_1^{(k)}} g_k'(x)\, dx \geqslant \varepsilon(x_1^{(k)} - x_0^{(k)}). \tag{3.13}$$

This leads to a contradiction when $k \to \infty$ since the left-hand side of (3.13) goes to zero while the right-hand side approaches $\varepsilon(x_1 - x_0) > 0$. ∎

We are now prepared to prove our main theorem on the convergence of sequences of generalized convex functions and their derivatives.

(3.14) THEOREM. *Let* $\{u_0, ..., u_{n-1}\}$ *be a complete* WT-*system of continuously differentiable functions on* (a, b) *with* $u_0 \equiv 1$, *and assume that* U *is joined and* U' *is perfect. If* $\{f_k\} \subset \mathrm{C}(u_0, ..., u_{n-1})$ *converges nearly to a continuous function* f *then* $f_k \to f$ *and* $f_k' \to f'$ *uniformly on compact subsets of* (a, b).

Proof. Note that by (2.1) $\{u_1', ..., u_{n-1}'\}$ is a WT-system and $f_k' \in \mathrm{C}(u_1', ..., u_{n-1}')$ for all k. By (3.5) $f \in \mathrm{C}(u_0, ..., u_{n-1})$, hence, (2.10) asserts that f is continuously differentiable. Moreover, (3.12) implies that $f_k' \to f'$ nearly, so we can take recourse to (3.11) to conclude that $\{f_k\}$ and $\{f_k'\}$ converge uniformly on compact subsets of (a, b) to f and f', respectively. ∎

4. EXAMPLES

In this section we present various examples of WT-systems that illustrate the ideas formulated in the foregoing material. We start with a definition.

(4.1) DEFINITION. U is called *nondegenerate* if none of its nontrivial elements vanish on open subintervals of the domain. U is called *endpoint nondegenerate* (END) if no element vanishes on an interval of the form (a, ξ) or (ξ, b).

T-spaces (spaces spanned by T-systems) and WT-spaces of analytic functions, such as polynomials, are examples of nondegenerate WT-spaces. END WT-spaces are easily constructed (e.g., by taking a T-system and replacing its values on some interior interval by those obtained with linear interpolation), however, such spaces seem to be rare in most applications.

(4.2) EXAMPLE. END WT-spaces are perfect. This is an immediate consequence of (1.10) and the proof of (1.15). An application of either (1.9) or (1.13) and the fact that every WT-space has a basis that is a complete WT-system [9] gives Theorem 4.3, first presented in [14].

(4.3) THEOREM. *If U is* END *then it has a positive element iff it has no vanishing points.*

We now turn our attention to spline and "spline-like" spaces. The basic building blocks for such spaces are the power functions x^j and the truncated power functions defined by

$$\begin{aligned}(x-\xi)_+^j &= (x-\xi)^j, & x \geqslant \xi \quad (j=1,\dots),\\ &= 0, & x < \xi \quad (j=1,\dots).\end{aligned}$$

For $j=0$, we define $(x-\xi)_+^0$ to be 1 if $x \geqslant \xi$ and zero otherwise.

In the next several examples ξ is an arbitrary point in (a, b).

(4.4) EXAMPLE. Let $u_0(x)=(x-\xi)_+$, $u_1(x)=(x-\xi)_+^2$. Since $u_0 \geqslant 0$, $\{u_0\}$ is a WT-system. A simple calculation shows that $\{u_0, u_1\}$ is also a WT-system; in fact for $x_0 < x_1$, $U\binom{0,1}{x_0,x_1} > 0$ iff $\xi < x_0 < x_1$. The system $\{u_0\}$ is clearly disconnected, as is $\{u_0, u_1\}$, but $\{u_0, u_1\}$ is joined since a point $\eta \in (a, b)$ is essential iff $\eta \in (\xi, b)$, and for all such points $\xi < x_0 < \eta < x_1 \Rightarrow \{x_0, \eta\}$ and $\{\eta, x_1\}$ are T-sets.

(4.5) EXAMPLE. Let $u_0(x)=(x-\xi)_+$ as before, and let $u_1(x)=(x-\xi)^2$. Then $\{u_0\}$ is a disconnected WT-system, and $\{u_0, u_1\}$ is a connected, disjoined WT-system. Indeed, direct computation of the determinant of the system shows that $U\binom{0,1}{x_0,x_1} > 0$ if $x_0 < \xi < x_1$ or $\xi < x_0 < x_1$, and we get zero otherwise. Any point $\eta \in (a, \xi)$ is essential, but $U\binom{0,1}{x_0,\eta} = 0$ for all $x_0 < \eta$, which shows that $\{u_0, u_1\}$ is disjoined.

(4.6) EXAMPLE. Let $u_0(x)=|x-\xi|$, $u_1(x)=(x-\xi)_+$. Then $\{u_0\}$ is a connected (in fact nondegenerate) WT-system, $\{u_0, u_1\}$ is a WT-system but is neither connected nor joined. Indeed, we have $U\binom{0,1}{x_0,x_1} > 0$ if $x_0 < \xi < x_1$ and $U\binom{0,1}{x_0,x_1} = 0$ otherwise, from which our claims readily follow.

(4.7) EXAMPLE. Let u_0 and u_1 be defined as in (4.6), and let $u_2(x) = (x-\xi)^2$. One checks that $U\binom{0,1,2}{x_0,x_1,x_2} \geqslant 0$, with strict inequality iff $x_0 < x_1 < \xi < x_2$ or $x_0 < \xi < x_1 < x_2$. Thus $\{u_0, u_1, u_2\}$ is both connected and joined, although $\{u_0, u_1\}$ is neither.

We now consider WT-spaces of spline functions. A polynomial spline of degree $k-1$ with simple knots $\xi_1 < \cdots < \xi_r$ is a function of the form

$$s(x) = \sum_{i=0}^{k-1} a_i x^i + \sum_{j=1}^{r} c_j (x-\xi_j)_+^{k-1}. \tag{4.8}$$

It is well known [1] that the basis

$$\{1, x, ..., x^{k-1}, (x-\xi_1)_+^{k-1}, ..., (x-\xi_r)_+^{k-1}\}, \tag{4.9}$$

for the space $S_{k,r}$ of functions of the form (4.8) is a WT-system. Further, $\{x_1, ..., x_{k+r}\}$ is a T-set for $S_{k,r}$ iff

$$x_i < \xi_i < x_{i+k} \qquad (i = 1 \text{ to } r). \tag{4.10}$$

(4.11) THEOREM. *The spline basis* (4.9) *is joined* (*and hence perfect*) *iff* $k \geqslant 3$.

Indeed, if $k=2$ then for $\xi = \xi_{l-1}$ $(2 \leqslant l \leqslant r+1)$, $\{x_1, ..., x_{r+2}\}$ is a T-set for $S_{2,r}$ only if $x_{l-1} < \xi < x_{l+1}$ so that by (1.18), $S_{2,r}$ is disjoined. For $k>2$ we note that since $x_i < x_{i+1} < x_{i+2} < x_{i+k}$, sets $\{x_1, ..., x_{k+r}\}$ satisfying (4.10) exist with $\xi_i = x_{i+1}$ or x_{i+2}, hence, $S_{k,r}$ is joined, at least as far as the ξ_i are concerned. However, one may easily check that a similar conclusion may be drawn regarding the other points.

A similar result is valid when multiplicities are allowed at the knots. A spline of degree $k-1$ with knots $\xi_1 < \cdots < \xi_r$ and corresponding multiplicities $\{m_i\}_{i=1}^r$ is a function of the form

$$s(x) = \sum_{i=0}^{k-1} a_i x^i + \sum_{i=1}^{r} \sum_{j=1}^{m_i} c_{ij} (x-\xi_i)_+^{k-j}. \tag{4.12}$$

We denote the space of such functions by $S_{k,\mathbf{m}}$, where $\mathbf{m} = (m_1, ..., m_r)$. The basis

$$\{1, x, ..., x^{k-1}, (x-\xi_1)_+^{k-m_1}, ..., (x-\xi_1)_+^{k-1}, ..., (x-\xi_r)_+^{k-m_r}, ..., (x-\xi_r)_+^{k-1}\}$$

is a WT-system, as Lemma 4.14 demonstrates.

(4.14) LEMMA [5]. *Let integers* $1 \leqslant m_i \leqslant k$ $(i = 1$ *to* $r)$ *be given and denote* $n = k + \sum_{i=1}^r m_i$. *Then for all* $a < \xi_1 < \cdots < \xi_r < b$ *and all* $a < x_1 < \cdots < x_n < b$,

$$U\begin{pmatrix} 0, ..., n-1 \\ x_1, x_2, ..., x_n \end{pmatrix} \geqslant 0,$$

with strict inequality iff $x_{l(i)} < \xi_i < x_{l(i-1)+k+1}$, *where* $\{u_0,\ldots, u_{n-1}\}$ denotes the functions (4.13), $l(i) = \sum_{j=1}^{i} m_j$ $(i = 1 \text{ to } r)$, $l(0) = 0$. *If for some* i, $m_i = k$ *then the determinant remains positive when* $\xi_i = x_{l(i-1)+k+1}$.

Suppose that u_n is generalized convex with respect to the system (4.13), that is $\{u_0,\ldots, u_n\}$ is a WT-system. Then, with the aid of (4.14) we may reason as in the case of simple knots and conclude as follows.

(4.15) THEOREM. *If* u_n *is generalized convex with respect to the functions* (4.13) *then for* $k \geqslant 2$, u_n *is continuous in* $(a, b)\backslash\{\xi_1,\ldots, \xi_r\}$. *If* $k > 2$ *then* u_n *is continuous at each* ξ_i *for which* $k - m_i \geqslant 2$; *if* $k - m_i \geqslant 2$ *for each* i *then* $S_{k,\mathbf{m}}$ *is joined.*

With regard to differentiability of u_n, we may apply (2.10) and derive

(4.16) THEOREM. *Under the same conditions as in* (4.15), $u_n^{(k-2)}$ *is continuous in* $(a, b)\backslash\{\xi_1,\ldots, \xi_r\}$ *and* $u_n^{(k-2-m_i)}$ *is continuous at* ξ_i $(i = 1$ *to* $r)$.

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